

# Quantum(-like) decision making: on validity of the Aumann theorem

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## Abstract

Through set-theoretic formalization of the notion of common knowledge, Aumann proved that if two agents have the common priors, and their posteriors for a given event are common knowledge, then their posteriors must be equal. In this paper we investigate the problem of validity of this theorem in the framework of quantum(-like) decision making.

## 1 Introduction

We remark that during recent years the mathematical formalism of quantum mechanics was widely applied to problems of decision making and more generally modeling of cognition, see, e.g., the monographs [1]–[4] as well as the series of articles [5]–[27]. This project is based on the *quantum-like paradigm* [2]: that information processing by complex cognitive systems (including social systems) taking into account contextual dependence of information and probabilistic reasoning can be mathematically described by quantum information and probability theories.

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One can find evidences of violation of laws of classical probability theory, e.g., in violation of the *law of total probability*. Its violation have been found in various sets of statistical data, see, e.g., [9], [11], [6],[10],[12], [15], [27], [2], [16], [4], [21]. The derivation of this law is based on the additivity of classical probability measures and the classical definition of conditional probabilities based on the *Bayes formula*. Thus the law of total probability can be violated as the result of violation of either additivity of classical probability, cf. with Feynman's viewpoint [31], or classical Bayesian rule or both jointly. One can say that this is an integral statistical test of classicality of probability combining its two basic features, additivity and Bayesianity. It is interesting to find cognitive phenomena in which just one of these factors is responsible for deviation from the classical probabilistic predictions.

The role of the Bayesian updating in decision making was analyzed in [23] with application to the problem of *human probability judgment errors*; in [26] this analysis was performed for such an important psychological phenomenon as *cognitive dissonance*. In both studies it was shown that by using quantum probability updating one can present consistent mathematical descriptions of aforementioned problems.

In this paper we show that the quantum generalization of the Bayesian updating leads to violation of the celebrating *Aumann theorem* [28], [29] which states that *if two agents have the common priors, and their posteriors for a given event  $E$  are common knowledge, then their posteriors must be equal; agents with the same priors cannot agree to disagree*. In this note we show that in *some contexts* agents using quantum(-like) information processing can *agree to disagree* even if they have the common priors, and their posteriors for a given event  $E$  are common knowledge. The most interesting problem is to find elements of classical Aumann's model for common knowledge whose quantum generalization induces violation of his theorem, we preset one of sufficient conditions of validity of the Aumann theorem even for agents whose information processing is described by quantum information theory and quantum probability.

## 2 Quantum(-like) approach to common knowledge

Following von Neumann [32] and Birkhoff and von Neumann [30] we represent *events, propositions*, as orthogonal projectors in complex Hilbert space  $H$ . Denote the scalar product in  $H$  as  $\langle \cdot | \cdot \rangle$ . For an orthogonal projector  $P$ , we set  $H_P = P(H)$ , its image, and vice versa, for subspace  $L$  of  $H$ , the corresponding

orthogonal projector is denoted by the symbol  $P_L$ .

The set of orthogonal projectors is a *lattice* with the order structure:  $P \leq Q$  iff  $H_P \subset H_Q$  or equivalently, for any  $\psi \in H$ ,  $\langle \psi | P \psi \rangle \leq \langle \psi | Q \psi \rangle$ . For a pure state  $|\psi\rangle$ , we set  $P_\psi = |\psi\rangle\langle\psi|$ , the orthogonal projector on this vector,  $P_\psi\phi = \langle\phi|\psi\rangle\psi$ .

Aumann's considerations [28], [29] are applicable for a finite number of *agents*, call them  $i = 1, 2, \dots, N$ . These individuals are about to learn the answers to various multi-choice *questions*, to make observations.

In our quantum-like model the “*states of the world*” are given by pure states. Questions posed by agents are mathematically described by self-adjoint operators, say  $A^{(i)}$ . We state again that events (propositions) are identified with orthogonal projectors. For the state of the world  $\psi$ , an event  $P$  occurs (takes place with probability 1) if  $\psi$  belongs to  $H_P$ .

To simplify considerations, we proceed in the case of the finite dimensional state space of the world,  $m = \dim H < \infty$ . Here each self-adjoint operator can be represented as a linear combination of orthogonal projectors to its eigen-subspaces. In particular, the questions of agents can be expressed as  $A^{(i)} = \sum_j a_j^{(i)} P_j^{(i)}$ , where  $(a_j^{(i)})$  are real numbers, all different eigenvalues of  $A^{(i)}$ , and  $(P_j^{(i)})$  are the orthogonal projectors onto the corresponding eigen-subspaces. Here  $(a_j)$  encode possible answers to the question of the  $i$ th agent. The system of projectors  $\mathcal{P}^{(i)} = (P_j^{(i)})$  is the spectral family of  $A^{(i)}$ . Hence, for any agent  $i$ , it is a “disjoint partition of unity”:

$$\bigvee_k P_k^{(i)} = I, \quad P_k^{(i)} \wedge P_m^{(i)} = 0, k \neq m. \quad (1)$$

We remark that (1) is simply the lattice-theoretical expression of the following operator equalities:

$$\sum_k P_k^{(i)} = I, \quad P_k^{(i)} P_m^{(i)} = 0, k \neq m. \quad (2)$$

This spectral family can be considered as information representation of the world by the  $i$ th agent. In particular, “getting the answer  $a_j^{(i)}$ ” is the event which is mathematically described by the projector  $P_j^{(i)}$ .

If the state of the world is represented by  $\psi$  and, for some  $k_0$ ,  $P_\psi \leq P_{k_0}^{(i)}$ , then

$$p_\psi(P_{k_0}^{(i)}) = \text{Tr} P_\psi P_{k_0}^{(i)} = 1 \text{ and, for } k \neq k_0, p_\psi(P_k^{(i)}) = \text{Tr} P_\psi P_k^{(i)} = 0.$$

Thus, in this case, the event  $P_{k_0}^{(i)}$  happens with the probability one and other events from information representation of the world by the  $i$ th agent have zero probability.

However, opposite to the classical case, in general  $\psi$  need not belong to any concrete subspace  $H_{P_k^{(i)}}$ . Nevertheless, for any pure state  $\psi$ , there exists the minimal projector  $Q_\psi^{(i)}$  of the form  $\sum_m P_{j_m}^{(i)}$  such that  $P_\psi \leq Q_\psi^{(i)}$ . Set  $O_\psi^{(i)} = \{j : P_j^{(i)}\psi \neq 0\}$ . Then  $Q_\psi^{(i)} = \sum_{j \in O_\psi^{(i)}} P_j^{(i)}$ . The projector  $Q_\psi^{(i)}$  represents the  $i$ th agent's knowledge about the  $\psi$ -world. We remark that  $p_\psi(Q_\psi^{(i)}) = 1$ .

Consider the system of projectors  $\tilde{\mathcal{P}}^{(i)}$  consisting of sums of the projectors from  $\mathcal{P}^{(i)}$  :

$$\tilde{\mathcal{P}}^{(i)} = \{P = \sum_m P_{j_m}^{(i)}\}. \quad (3)$$

Then

$$Q_\psi^{(i)} = \min\{P \in \tilde{\mathcal{P}}^{(i)} : P_\psi \leq P\}. \quad (4)$$

**Definition 1.** For the  $\psi$ -state of the world and the event  $E$ , the  $i$ th agent knows  $E$  if

$$Q_\psi^{(i)} \leq E. \quad (5)$$

It is evident that if, for the state of the world  $\psi$ , the  $i$ th agent knows  $E$ , then  $\psi \in H_E$ . In general the latter does not imply that  $E$  is known (for the state  $\psi$ ). However, if  $\psi \in E = P_j^{(i)}$ , then this event is known for  $i$ . The same is valid for any event of the form  $E = P_{j_1}^{(i)} \vee \dots \vee P_{j_k}^{(i)} (= P_{j_1}^{(i)} + \dots + P_{j_k}^{(i)})$ ; if  $\psi \in H_E$ , then such  $E$  is known for  $i$ .

We remark that the straightforward analog of the classical definition would be based on condition  $P_j^{(i)} \leq E$  for  $P_\psi \leq P_j^{(i)}$ , instead of more general condition (5). However, it would trivialize the class of possible states of the world.

We now define the *knowledge operator*  $K_i$  which applied to any event  $E$ , yields the event “ $i$ th agent knows that  $E$ .”

**Definition 2.**  $K_i E = P_{H_{K_i E}}$ , where  $H_{K_i E} = \{\phi : Q_{\phi/\|\phi\|}^{(i)} \leq E\}$ .

**Proposition 1.** For any event  $E$ , the set  $H_{K_i E}$  is a linear subspace of  $H$ .

**Proof.** Take two vectors  $\phi_1, \phi_2 \in H_{K_i E}$  and consider their linear combination  $\phi = a_1\phi_1 + a_2\phi_2$ . We consider also the corresponding pure states  $\psi_1 = \phi_1/\|\phi_1\|, \psi_2 = \phi_2/\|\phi_2\|$  and  $\psi = \phi/\|\phi\|$ . We have  $Q_{\psi_m}^{(i)} \leq E$ . Thus  $\psi_m = \sum_{j \in O_{\psi_m}^{(i)}} P_j^{(i)}\psi_m$ . It is clear that  $\phi$  can be represented in the form  $\phi = \sum_{j \in O_{\phi_1}^{(i)} \cup O_{\phi_2}^{(i)}} P_j^{(i)}\psi$ . Therefore  $O_\psi^{(i)} \subset O_{\psi_1}^{(i)} \cup O_{\psi_2}^{(i)}$  and, hence,  $Q_\psi^{(i)} \leq E$ .

Thus definition 2 is consistent. The operator  $K_i$  has the properties similar to the properties of the classical knowledge operator:

Now, as in the classical case, we define:

$$M_0E = E, M_1E = K_1E \wedge \dots \wedge K_NE, \dots, M_{n+1}E = K_1M_nE \wedge \dots \wedge K_NM_nE, \dots$$

As usual,  $M_1E$  is the event “all agents know that  $E$ ” and so on. We can rewrite this definition by using subspaces, instead of projectors:

$$H_{M_1E} = H_{K_1E} \cap \dots \cap H_{K_NE}, \dots, H_{M_{n+1}E} = H_{K_1M_nE} \cap \dots \cap H_{K_NM_nE}, \dots$$

Now we define the “*common knowledge*” operator, as mutual knowledge of all finite degrees:

$$\kappa E = \bigwedge_{n=0}^{\infty} M_n E.$$

As in the classical case we have that “Where something is common knowledge everybody knows it.”

**Lemma 1.** *If  $\kappa E \neq 0$ , then, for each  $i$ , it can be represented as*

$$\kappa E = \sum_m P_{j_m}^{(i)}. \quad (6)$$

**Proof.** Take any nonzero vector  $\phi \in H_{\kappa E}$ . Then it belongs to  $H_{K_i M_n E}$  for any  $n$ . Thus  $Q_{\phi/\|\phi\|}^{(i)} \leq M_n E$  and, hence,  $Q_{\phi/\|\phi\|}^{(i)} \leq \bigwedge_{n=0}^{\infty} M_n E = \kappa E$ . Hence, for each  $\phi \in H_{\kappa E}$ , we have  $\phi = Q_{\phi/\|\phi\|}^{(i)} \phi$ . Thus (6) holds.

### 3 Quantum(-like) viewpoint on the Aumann’s theorem

#### 3.1 Common prior assumption

Suppose now that both agents assigned to possible states of the world the same quantum probability distribution given by the density operator  $\rho$ , a priori state. Thus they do not know exactly the real state of the world (the latter is always a pure state) and a possible state of the world appears for them as a mixed quantum state. A priori probability for possible states of the world is combined with the information pictures used by the agents and given by their partitions of unity.

#### 3.2 Quantum probability update

Consider some event  $E$ . The agents assign to it probabilities after conditioning  $\rho$  on the answers to their questions (on their information representations

of the world):

$$q_k^{(i)} = p_\rho(E|P_k^{(i)}) = \frac{\text{Tr}P_k^{(i)}\rho P_k^{(i)}E}{\text{Tr}P_k^{(i)}\rho P_k^{(i)}}. \quad (7)$$

We remark that the agents can assign probabilities conditioned on the results of observations only for the answers  $a_k^{(i)}$  such that  $\text{Tr}P_k^{(i)}\rho P_k^{(i)} > 0$ .

Consider the events

$$C_{q^{(i)}} \equiv \{q_k^{(i)} = q^{(i)}\} = \bigvee_{\{k:q_k^{(i)}=q^{(i)}\}} P_k^{(i)}, \quad (8)$$

$i = 1, \dots, N$ , and set

$$C_{q^{(1)}\dots q^{(N)}} = \{q_k^{(1)} = q^{(1)}, \dots, q_k^{(N)} = q^{(N)}\} = \bigwedge_i C_{q^{(i)}}.$$

**Remark.** Consider the classical Aumann model [28], [29]. Here

$$q^{(i)}(\omega) = p(E|P^{(i)}(\omega)) = \frac{p(E \cap P^{(i)}(\omega))}{p(P^{(i)}(\omega))} \quad (9)$$

and  $C_{q^{(i)}} \equiv \{\omega : q_k^{(i)}(\omega) = q^{(i)}\}$ . We remark that if for some  $\omega_0$  the probability  $q^{(i)}(\omega_0) = q^{(i)}$ , then, for any  $\omega \in P^{(i)}(\omega_0)$ , the probability  $q^{(i)}(\omega) = q^{(i)}$ . Thus

$$C_{q^{(i)}} = \bigcup_{\{k:q_k^{(i)}=q^{(i)}\}} P_k^{(i)}, \quad (10)$$

cf. (8).

We remark that, in fact, as a consequence of mutual orthogonality of projectors from the spectral family of any Hermitian operator, the event  $C_{q^{(i)}}$  can be represented as

$$C_{q^{(i)}} = \sum_{\{k:q_k^{(i)}=q^{(i)}\}} P_k^{(i)}.$$

Thus the event  $C_{q^{(1)}\dots q^{(N)}}$  has representation:

$$C_{q^{(1)}\dots q^{(N)}} = \left( \sum_{\{k:q_k^{(1)}=q^{(1)}\}} P_k^{(1)} \right) \wedge \dots \wedge \left( \sum_{\{k:q_k^{(N)}=q^{(N)}\}} P_k^{(1)} \right).$$

### 3.3 Interference prevents agreement

Suppose that the possibility of  $C_{q^{(1)} \dots q^{(N)}}$  becoming common knowledge is not ruled out completely, i.e.,

$$p_\rho(\kappa C_{q^{(1)} \dots q^{(N)}}) > 0. \quad (11)$$

Then the straightforward quantum generalization of the classical Aumann theorem [28], [29] would imply that  $q^{(1)} = \dots = q^{(N)}$ . However, this is not the case! (as it may be expected, since the classical Aumann theorem was heavily based on usage of Boolean logics).

By Lemma 1 the common knowledge projector can be represented as  $\kappa E = \sum_j P_{k_j}^{(i)}, i = 1, \dots, N$ . For each such  $P_{k_j}^{(1)}, \dots, P_{k_j}^{(N)}$ , we have

$$p_\rho(E|P_{k_j}^{(1)}) = q^{(1)}, \dots, p_\rho(E|P_{k_j}^{(N)}) = q^{(N)}.$$

(In particular, for any such projector conditional probabilities are well defined, i.e.,  $\text{Tr} P_{k_j}^{(i)} \rho P_{k_j}^{(i)} > 0$ .) Consider now the conditional probability:

$$p_\rho(E|\kappa C_{q^{(1)} \dots q^{(N)}}) = \frac{\text{Tr} \kappa C_{q^{(1)} \dots q^{(N)}} \rho \kappa C_{q^{(1)} \dots q^{(N)}} E}{\text{Tr} \kappa C_{q^{(1)} \dots q^{(N)}} \rho \kappa C_{q^{(1)} \dots q^{(N)}}}.$$

First we remark that, for any projector  $M$ ,  $\text{Tr} M \rho M = \text{Tr} \rho M$ . Thus

$$p_\rho(E|\kappa C_{q^{(1)} \dots q^{(N)}}) = \frac{\text{Tr} \kappa C_{q^{(1)} \dots q^{(N)}} \rho \kappa C_{q^{(1)} \dots q^{(N)}} E}{\text{Tr} \rho \kappa C_{q^{(1)} \dots q^{(N)}}}.$$

By using representation given by Lemma 1 we obtain

$$p_\rho(E|\kappa C_{q^{(1)} \dots q^{(N)}}) = \frac{1}{\text{Tr} \rho \kappa C_{q^{(1)} \dots q^{(N)}}} \left( \sum_j \text{Tr} P_{k_j}^{(i)} \rho P_{k_j}^{(i)} E + \sum_{j \neq m} \text{Tr} P_{k_j}^{(i)} \rho P_{k_m}^{(i)} E \right). \quad (12)$$

The first (diagonal) sum can be written as

$$\frac{1}{\text{Tr} \rho \kappa C_{q^{(1)} \dots q^{(N)}}} \sum_j \frac{\text{Tr} P_{k_j}^{(i)} \rho P_{k_j}^{(i)} E}{\text{Tr} \rho P_{k_j}^{(i)}} \text{Tr} \rho P_{k_j}^{(i)} = \frac{q_i}{\text{Tr} \rho \kappa C_{q^{(1)} \dots q^{(N)}}} \text{Tr} \sum \rho P_{k_j}^{(i)} = q^{(i)}.$$

In the absence of the off-diagonal term in (12) we get  $q^{(1)} = \dots = q^{(N)}$ . This corresponds to the classical case. However, in general the off-diagonal term does not vanish – this is *the interference type effect*.

Hence, in general *the Aumann theorem is not valid for “quantum(-like) decision makers*. Thus *agents processing information in the quantum information and probability framework can agree on disagree*.

### 3.4 Sufficient conditions of validity of the quantum(-like) version of the Aumann theorem

Now we present a special situation in which even quantumly thinking agents cannot agree on disagree.

**Proposition 2.** *Let the common a priori state  $\rho$  is given by the unity operator normalized by the dimension. If condition (11), the assumption of common prior, holds, then  $q^{(1)} = \dots = q^{(N)}$ .*

To prove this statement, we point to the fact that, for such  $\rho$ , the off-diagonal term in (12) equals to zero and the proof can be completed in the same way as in the classical case.

If  $\rho = I/\dim H$ , then all states of the world are equally possible. Thus, for all agents, the a priori state of the world was gained in the total absence of information about the world. In this case these agents have to come to the same posteriors (if their posteriors are common knowledge), though they may base their posteriors on different information: the information partitions of unity, see (1), can be incompatible, i.e., the projectors  $(P_j^{(i)})$  need not commute with the projectors  $(P_j^{(s)})$ .

We remark that, although in the rigorous mathematical framework a density operator cannot be scaling of the unit operator, formally one can operate with such “generalized density operators”, see von Neumann [32]. Thus formally Proposition 2 can be generalized to infinite-dimensional state spaces.

In fact, Proposition 2 is a special case of a more general statement which will be soon formulated. However, we started with Proposition 2, since it has the very clear interpretation. The interpretation of the following statement is not straightforward:

**Theorem 1.** *Let the common a priori state  $\rho$  commutes with the elements of all partitions, i.e., for  $i = 1, \dots, N$ ,*

$$[\rho, P_j^{(i)}] = 0 \tag{13}$$

*for any  $j$ . If condition (11) holds, then  $q^{(1)} = \dots = q^{(N)}$ .*

Here we again see that the interference term in (12) equals to zero.

As was mentioned, the interpretation of the basic condition of this theorem is not straightforward. In quantum mechanics, commutativity of observables is interpreted as the condition of joint measuring. However, commutativity of an observable and a quantum state has no direct interpretation.



**Lemma 3.** *Let  $\rho$  have non-degenerate spectrum and let the condition (13) holds. Then the partitions  $(P_j^{(1)}), \dots, (P_j^{(N)})$  are compatible, i.e.,  $[P_j^{(i)}, P_m^{(s)}] = 0$  for any pair  $j, m$  and  $i, s$ .*

**Proof.** Suppose that  $\rho$  has non-degenerate spectrum, i.e.,  $\rho = \sum_k p_k P_{e_k}$ , where  $p_k \neq p_m, k \neq m$ , and  $(e_k)$  is the orthonormal basis consisting of eigenvectors of  $\rho$ . Then, for any orthogonal projector  $P$ , the condition  $[P, \rho] = 0$  implies that there exists a set of indexes  $O_P$  such that  $P = \sum_{m \in O_P} P_{e_m}$ . This is easy to show. We have  $\rho P = P \rho$ , i.e., for any pair of basis vectors  $e_t, e_s$ , we have, on one hand,  $\langle e_t | \rho P | e_s \rangle = p_t \langle e_t | P | e_s \rangle$  and, on the other hand  $\langle e_t | P \rho | e_s \rangle = p_s \langle e_t | P | e_s \rangle$ . Hence, for  $t \neq s$ ,  $\langle e_t | P | e_s \rangle = 0$ . Thus  $P | e_s \rangle = a_s | e_s \rangle$ , where  $a = 0, 1$ . Set  $O_P = \{s : a_s = 1\}$ . For the projectors  $P_j^{(i)}$ , such sets will be denoted as  $O_j^{(i)}$ .

Thus in the non-degenerate case the condition (13) implies that  $P_j^{(i)} = \sum_{k \in O_j^{(i)}} P_{e_k}$ . Then  $P_j^{(i)} P_m^{(s)} = \sum_{k \in O_j^{(i)} \cap O_m^{(s)}} P_{e_k} = P_m^{(s)} P_j^{(i)}$ . Thus the (quantum) information partitions are compatible.

However, if the spectrum of the state  $\rho$  is degenerate, then the condition (13) does not imply compatibility of partitions of two agents (see Proposition 2).

**Corollary 1.** *Even in the case of incompatible (quantum) information partitions, it is possible to find such common a priori (quantum) states that it is impossible to agree on disagreeing.*

## 4 Conclusion

Agents representing and processing information in the quantum(-like) manner can agree on disagree. Thus our quantum(-like) model of probability update in the presence of common knowledge matches better with the real situation.

Typically in classical analysis of sources of violations of the Aumann theorem (and it is often violated in reality) the common a priori probability distribution and the presence of common knowledge are pointed as questionable assumptions in Aumann's argumentation. We show that the validity of these assumptions does not prevent from the possibility that agents agree on disagree.

The main conclusion is that agents can simply use more general rules for processing of information and probability than given by the classical set-measure-theoretic model based on the Kolmogorov axiomatics of probability theory. And this model has its own restricted domain of applications, as

any mathematical model, cf. with the Euclidean model of geometry, and departure from it given by Lobachevsky geometry (which plays an important role in special relativity theory).

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